13.
Nonparametric estimation of nonlinear characteristic in Hammerstein system

0.1. Kernel method

The kernel regression estimate has the form

$$\widehat{\mu}_N(u) = \frac{\sum_{k=1}^{N} y_k K\left(\frac{u_k - u}{h_N}\right)}{\sum_{k=1}^{N} K\left(\frac{u_k - u}{h_N}\right)}, \tag{1}$$

where h_N is a bandwidth parameter, which fulfils the following conditions

$$h_N \to 0 \text{ and } Nh_N \to \infty, \text{ as } N \to \infty, \tag{2}$$

and $K()$ is a kernel function, such that

$$K(x) \geqslant 0, \ \sup K(x) < \infty \text{ and } \int |K(x)|\, dx < \infty. \tag{3}$$

Standard examples are $K(x) = I_{[-0.5,0.5]}(x) \triangleq \begin{cases} 1, \text{ as } |x| \leqslant 0.5 \\ 0, \text{elsewhere} \end{cases}$, $(1 - |x|)I_{[-1,1]}(x)$ or $\left(1/\sqrt{2\pi}\right) e^{-x^2/2}$ and $h_N = h_0 N^{-\alpha}$ with $0 < \alpha < 1$ and positive $h_0 = \text{const}$.

Remark 1 *It holds that*

$$\widehat{\mu}_N(u) \to \mu(u) \text{ in probability, as } N \to \infty, \tag{4}$$

at every $u \in Cont(\mu, f)$, the set of continuity points of $\mu(u)$ and $f(u)$, at which $f(u) > 0$. If moreover $\mu(u)$ and $f(u)$ are at least two times continuously differentiable at u, then for $h_N = h_0 N^{-1/5}$ the convergence rate is

$$|\widehat{\mu}_N(u) - \mu(u)| = O(N^{-2/5}) \text{ in probability.} \tag{5}$$

0.2. Orthogonal series expansion method

Denoting $g(u) \triangleq \mu(u)f(u)$ one can write $\mu(u) = g(u)/f(u)$. Let $\{\varphi_i(u)\}_{i=0}^{\infty}$ be the complete set of orthonormal functions in the input domain. If $g()$ and $f()$ are square integrable, then $g(u) = \sum_{i=0}^{\infty} \alpha_i \varphi_i(u)$, $f(u) = \sum_{i=0}^{\infty} \beta_i \varphi_i(u)$, where $\alpha_i = E y_k \varphi_i(u_k)$ and $\beta_i = E\varphi_i(u_k)$, is an orthogonal series representation of $g(u)$ and $f(u)$ in the basis $\{\varphi_i(u)\}_{i=0}^{\infty}$. The standard estimates of the coefficients α_i's and β_i's are

$$\widehat{\alpha}_{i,N} = \frac{1}{N}\sum_{k=1}^{N} y_k \varphi_i(u_k), \qquad \widehat{\beta}_{i,N} = \frac{1}{N}\sum_{k=1}^{N} \varphi_i(u_k),$$

which leads to the following ratio estimate of $\mu(u)$

$$\widehat{\mu}_N(u) = \frac{\sum_{i=0}^{q(N)} \widehat{\alpha}_{i,N}\varphi_i(u)}{\sum_{i=0}^{q(N)} \widehat{\beta}_{i,N}\varphi_i(u)}, \tag{6}$$

where $q(N)$ is some cut-off level.

Remark 2 *To assure vanishing of the approximation error, the scale* $q(N)$ *must behave so that* $\lim_{N\to\infty} q(N) = \infty$. *For the convergence of* $\widehat{\mu}_N(u)$ *to* $\mu(u)$, *the rate of* $q(N)$*-increasing must be appropriately slow, e.g.,* $\lim_{N\to\infty} q^2(N)/N = 0$ *for trigonometric or Legendre series,* $\lim_{N\to\infty} q^6(N)/N = 0$ *for Laguerre series,* $\lim_{N\to\infty} q^{5/3}(N)/N = 0$ *for Hermite series.*